3. Money in the Utility Function

Slides based on G. Illing, University of Munich

Can we provide a microfoundation for Cagan money demand?

Important for

- understanding the determinants of money demand
- relevance of bubble solutions
- integrating money in general equilibrium models
- welfare evaluation of monetary policy
3.1. Money in the Utility Function: Brock/Sidrauski Model

Extension of Standard Growth Models to include money:

Sidrauski 1967; Brock 1974 (see: Blanchard /Fischer, chapter 4/5; Woodford chapter 2; Walsh, chapter 2)

Money as one of many assets (some financial, some real).
Nominal return on money $i_{m,t}$ is dominated by the return $i_t$ on safe bonds.
Money is useful as it provides an indirect utility stemming from:
- liquidity allows to carry out transactions
- and to save time compared to being illiquid

Per period payoff function: $U(C, m)$

Shortcut to model liquidity services (indirect utility, derived from transaction costs (shopping time models); cash in advance constraint; OLG)

$U(C, m)$ is assumed to be concave and increasing in both arguments; $C \geq 0; \ m = M/P \geq 0$;

$U_c > 0; \ U_m \geq 0$
Frequently assumed: **additive separability**: \( U(C, m) = u(C) + V(m) \)

Frequently assumed: there is some **point of satiation** \( m^* \) for real money balances: satiation level \( m^* \) with \( U_m(C, m^*) = 0 \) and \( U_m(C, m) < 0 \) for \( m > m^* \).

Important property of utility function: money is **"essential"** if \( \lim_{m \to 0} m U_m(m) > 0 \).
Representative Household maximises: \[ \text{Max } u = \sum_{t=0}^{T} \beta^t U(C_t, \frac{M_t}{P_t}) \]

subject to
(1) Intertemporal Budget constraint
(2) endpoint constraints; No-Ponzi game constraint

Key Task: Formulate the relevant budget constraint with complete set of nominal assets; perfect foresight; pure endowment economy.

Initial wealth \( W_0 \) given, endowment stream of nominal net income \( P_t(Y_t - T_t) \) given.
Choose \( C_t, M_t/P_t \) and the allocation of nominal and real assets \( B_t, K_t \) for a given sequence of intertemporal prices \( \{P_t\} \) and \( \{Q_{t,t+s}\} \).

The riskless short term nominal interest rate is \( i_t \):

\[ \frac{1}{(1 + i_t)} = Q_{t,t+1} \]

**Nominal discount factor** \( Q_{t,T} = \prod_{v=t+1}^{T} Q_{v-1,v} = \prod_{v=t}^{T-1} \frac{1}{(1 + i_v)} \) (define: \( Q_{t,t} = 1 \))
(1) Intertemporal Budget constraint

In discrete time analysis different timing conventions for stocks and flows. Here: the notation of Woodford (2003)\(^1\)

**Discrete Time Analysis: Timing convention**

\[
W_t = (1 + i_{t-1}) B_{t-1} + M_{t-1} \quad \quad \quad W_{t+1} = (1 + i_t) B_t + M_t
\]

- Given Wealth \(W_t\) and Income \(Y_t\): Consumption choice and \(M_t/B_t\) choice of portfolio allocation
- Money balances at the end of period \(t\): \(M_t\)
- Bonds bought at face value at \(t\), return \(i_t\) is increasing nominal wealth \(W_{t+1}\)
- Budget Constraint at \(t\): \(B_t + M_t + P_t C_t = P_t Y_t + W_t\)
  \[\frac{1}{1+i_t} W_{t+1} + \frac{i_t}{1+i_t} M_t + P_t C_t = P_t Y_t + W_t\]

\(^1\) In an alternative timing convention, financial markets open at the beginning of period \(t\), and agents have to choose \(M_t\), \(B_t\) and \(K_t\). Then, agents receive income \(Y_t\) and can consume from liquid assets. Then, money balances at the end of period \(t\) are \(M_t + P_t (Y_t - T_t) - P_t C_t\). So here, money \(M_t\) providing liquidity is not the money held at the end of the period. How does this alternative convention affect the results? (see Woodford, Appendix A 16).
Including interest on money and real capital, define the beginning-of-period wealth as

\[ W_t = (1 + i_{t-1}) B_{t-1} + (1 + i^m_{t-1}) M_{t-1} + (1 + r_{t-1}) P_t K_{t-1} \]

- \( i_t \) nominal return for riskless one year bonds \( B_t \) held between \( t \) and \( t+1 \).
- \( r_t \) return for real bonds \( K_t \).
- \( i^m_t \) interest on money \( M_t \).

Relevant state variable: total wealth \( W_t \) given at \( t \).

Budget Constraint at \( t \):

\[ B_t + M_t + P_t C_t + P_t K_t \leq P_t(Y_t - T_t) + W_t \]

Use definition of wealth \( (W_{t+1}) \), solve for \( B_t \) and insert for \( B_t \) in period-\( t \) budget constraint to formulate the per period budget constraint \( t=1,\ldots,T \) as the evolution of nominal wealth:

\[
W_t = (1 + i_{t-1}) B_{t-1} + (1 + i^m_{t-1}) M_{t-1} + (1 + r_{t-1}) P_t K_{t-1}
\]

\[
W_{t+1} = (1 + i_t) B_t + (1 + i^m_t) M_t + (1 + r_t) P_{t+1} K_t
\]

\[
(1 + i_t) B_t = W_{t+1} - (1 + i^m_t) M_t - (1 + r_t) P_{t+1} K_t
\]

Solve for \( B_t \) and replace \( B_t \) in Budget constraint:
\[ B_i + M_i + P_i C_i + P_i K_i \leq P_i (Y_i - T_i) + W_i \]
\[ \iff \frac{W_{t+1} - (1 + t^m_i) M_i - (1 + r_i) P_{i+1} K_i}{1 + i_t} + M_i + P_i C_i + P_i K_i \leq P_i (Y_i - T_i) + W_i \]
\[ \iff \]
\[ P_t C_t + \frac{i_t - i_m}{1 + i_t} M_t + \frac{1}{1 + i_t} W_{t+1} + (P_t - \frac{1 + r_t}{1 + i_t} P_{t+1}) K_t \leq P_t (Y_t - T_t) + W_t \]

(2) Endpoint constraints

2a) \( W_0 \) given.

2b) \( W_{T+1} \geq 0 \) Solvency constraint: at the terminal date, assets left cannot be negative.
   Or “no Ponzi Game” constraint \( \lim_{T \to \infty} Q_{0,T+1} W_{T+1} \geq 0 \)

For \( T \to \infty, \lim_{T \to \infty} E_t (Q_{t,T} W_T) \geq 0 \) implies:

\[ W_{t+1} \geq - \sum_{T=t+1}^{\infty} E_{t+1} (Q_{t+1,T} (p_T y_T - T_T)) \]
\[ \iff -W_{t+1} \leq \sum_{T=t+1}^{\infty} E_{t+1} (Q_{t+1,T} (p_T y_T - T_T)) \]

Current debt cannot exceed present value of future net income.
Solve the optimisation problem using the standard Lagrangian approach:

\[
\begin{align*}
    u &= \sum_{t=0}^{T} \beta^t U(C_t, \frac{M_t}{P_t}) + \sum_{t=0}^{T} \lambda_t [P_t Y_t + W_t - P_t C_t - Q_{t,t+1} W_{t+1} - \frac{i_t - i_{m_t}}{1+i_t} M_t - (P_t - \frac{1+r_t}{1+i_t} P_{t+1}) K_t] \\
    + \tilde{\lambda}_0 W_0 + \lambda_{T+1} W_{T+1}
\end{align*}
\]

\( \lambda_t \) is the present-value shadow price.

This gives as FOCs:

a) \((C_t)\)  \( \beta^t U_c(C_t, \frac{M_t}{P_t}) = \lambda_t P_t \) \( \forall t \)

b) \((W_{t+1})\)  \( \lambda_t Q_{t,t+1} = \lambda_{t+1} \) \( \forall t \)  \( \rightarrow \) Arbitrage equation

c) \((K_t)\)  \( 1 + i_t = (1 + r_t) \frac{P_{t+1}}{P_t} \) \( \forall t \)  Fisher equation

otherwise: either \( B_t = 0 \) or \( K_t = 0 \) \( [B_t \geq 0; K_t \geq 0!] \)

d) \((M_t)\)  \( U_m(C_t, M_t / P_t) \beta^t \frac{1}{P_t} = \lambda_t \frac{i_t - i_{m_t}}{1+i_t} \) \( \forall t \)  \( \rightarrow \) Money demand (LM curve)

e) Transversality condition:  \( \lambda_{T+1} W_{T+1} = \lambda_0 Q_{0,T+1} W_{T+1} = 0 \) or \( \lim_{T \to \infty} E_t(Q_{0,T+1} W_{T+1}) = 0 \)
(a), (b) give:

\[
\frac{\beta^{t+1} U_c(C_{t+1}, M_{t+1} / P_{t+1})}{\beta^t U_c(C_t, M_t / P_t)} = Q_{t,t+1} \frac{P_{t+1}}{P_t}
\]

or, using (c):

\[
\frac{U_c(C_{t+1}, \frac{M_{t+1}}{P_{t+1}})}{U_c(C_t, \frac{M_t}{P_t})} = \frac{1 + \rho}{1 + i_t} \cdot \frac{P_{t+1}}{P_t} = \frac{1 + \rho}{1 + r_t}
\]

\[
\beta = \frac{1}{1 + \rho},
\]

\[
\rho = \text{rate of time preference}
\]

\[
U_m(C_t, \frac{M_t}{P_t}) = i_t - i_{m_t}
\]

\[
U_c(C_t, \frac{M_t}{P_t}) = 1 + i_t \quad \text{Money demand curve}
\]

**Interpretation of FOC conditions:**

(3) MRS between consumption today and tomorrow equal to real discount factor

\[
-\frac{d C_t}{d C_{t+1}} = 1 \cdot \frac{U_c(C_{t+1}, \frac{M_{t+1}}{P_{t+1}})}{1 + \rho} \cdot \frac{1}{U_c(C_t, \frac{M_t}{P_t})} = \frac{1}{1 + r_t}
\]
(4) MRS between consumption today and consumption of real money balances equal to opportunity cost of holding money balances (present value of interest lost next period due to holding money):

\[
- \frac{d C_t}{d M_t / P_t} = \frac{U_m(C_t; \frac{M_t}{P_t})}{U_c(C_t; \frac{M_t}{P_t})} = \frac{i_t - i_{mt}}{1 + i_t}
\]

(4)

Micro-based version of the LM curve

From FOC for optimal real balances, we can derive a money demand function which, in general, also depends on the consumption level, because of \( U'(c(t)) \).
Alternative Derivation of FOC conditions:
Integrate intertemporal budget constraints to one constraint in the Lagrangian.
(we ignore real capital, assume constant rate of inflation, constant real interest rates and no interest
on money holding): (1)+(2a) can be integrated towards:

\[
\sum_{t=0}^{T} Q_{0,t} \left( P_t C_t + \frac{i}{1+i} M_t \right) + Q_{0,T+1} W_{T+1} \leq W_0 + \sum_{t=0}^{T} P_t (Y_t - T_t) Q_{0,t}
\]

With constant nominal interest rates, we get:

\[
\sum_{t=0}^{T} \frac{1}{(1+i)^t} \left( P_t C_t + \frac{i}{1+i} M_t \right) + \frac{1}{(1+i)^T} W_{T+1} \leq W_0 + \sum_{t=0}^{T} P_t (Y_t - T_t) \frac{1}{(1+i)^t}
\]

Dividing by \( P_0 \), get the wealth constraint in real terms (use \( P_t = (1+\pi)^t P_0 \)):

\[
\sum_{t=0}^{T} \frac{1}{(1+r)^t} \left( C_t + \frac{i}{1+i} M_t \right) + \frac{1}{(1+r)^T} \frac{W_{T+1}}{P_0} \leq W_0 + \sum_{t=0}^{T} \frac{1}{(1+r)^t} (Y_t - T_t)
\]

Using (2b), this is equivalent to

\[
\sum_{t=0}^{T} \frac{1}{(1+r)^t} \left( C_t + \frac{i}{1+i} M_t \right) - \frac{W_0}{P_0} - \sum_{t=0}^{T} \frac{1}{(1+r)^t} (Y_t - T_t) \leq 0
\]

So reduce problem to a constrained static optimisation problem with the Lagrangian:

\[
L = \sum_{t=0}^{T} \beta^t U(C_t, \frac{M_t}{P_t}) - \lambda \left\{ \sum_{t=0}^{T} \frac{1}{(1+r)^t} \left( C_t + \frac{i}{1+i} M_t \right) - \frac{W_0}{P_0} - \sum_{t=1}^{T} \frac{1}{(1+r)^t} (Y_t - T_t) \right\}
\]
This gives as FOC:

(a) $\beta^t u_c(C_t) = \lambda \frac{1}{(1+r)^t}$

(b) $u_{M/P}(c_i; \frac{M_t}{p_t}) \frac{1}{(1+\rho)^t} = \lambda \frac{i}{1+i} \frac{1}{(1+r)^t}$

(c) Fisher equation (when introducing real assets)

Complementary slackness condition:

(d) $\lambda \left\{ \sum_{t=0}^{T} \frac{1}{(1+r)^t} \left( C_t + \frac{i}{1+i} \frac{M_t}{p_t} \right) - \frac{W_0}{P_0} - \sum_{t=1}^{T} \frac{1}{(1+r)^t} (Y_t - T_t) \right\} = 0$

For any interesting problem, $\lambda = u'(C_0) > 0$, so the wealth constraint must be binding. Thus, we get as necessary condition the transversality condition: $\frac{1}{(1+r)^T} \frac{W_{T+1}}{P_T} = 0$

(transversality condition ~ No Ponzi game constraint must be binding)
Model the frictions which motivate indirect utility of money: → Cash in advance and shopping time models

3.2. Cash in advance constraint

Extreme transaction technology: Some goods - cash goods $X_t$ - can be bought only by paying in cash: $P_t X_t \leq M_t$.

→ Consumption of cash goods is taxed by $\frac{i_t - i_m}{1 + i_t}$

Constraint is a special case of the MIU approach.
Representative household has no direct utility from money holding. She maximises:

$$\text{Max } \sum_{t=0}^{\infty} \frac{1}{(1+\rho)^t} U(C_t, X_t) \text{ s. t.}$$

(1) Flow budget constraint:

$$P_t C_t + P_t X_t + \frac{i_t - i_m}{1 + i_t} M_t + \frac{1}{1 + i_t} W_{t+1} + (P_t - \frac{1 + r_t}{1 + i_t} P_{t+1}) K_t \leq P_t (Y_t - T_t) + W_t$$

(2) Cash in advance constraint for cash-goods $P_t X_t \leq M_t$ for all $t$. 
Since cash holdings are costly, the second constraint will hold with equality. Thus, we can replace \( X_t = M_t / P_t \) in the utility function \( U(C_t, X_t) \) giving us a direct utility for real balances.

Substitute \( X_t = m_t = M_t / P_t \) in the objective function: optimization problem formally equivalent to max (indirect) utility of real money and credit goods.

But the functional form of the indirect utility function may be at odds with it being additive separable. We need to assume that it is additively separable w.r.t. cash goods and non-cash goods.

The property that money is essential \( \lim_{m \to 0} m U_m(m) > 0 \) now requires that cash goods are essential.
3.3. Shopping time model

$T(c_t, m_t)$ transaction technology: real resources used up when $c_t$ is consumed with real money balances $m_t$ available.

Money reduces transaction costs (shopping time required for purchasing goods).

$T(c_t, m_t)$ with $T_{c_t} > 0$; $T_{m_t} < 0$

Transform so as to maximize indirect utility function $V(x_t, m_t) = U(f(x_t, m_t))$.

Consider continuous time problem:

$$\max_{c(t); m(t)} V(0) = \int_{0}^{\infty} u[c(t); 1 - n(t)] e^{- \rho t} dt,$$

where $n(t)$ is fraction of time devoted to labor and shopping.

Normalize real wage to 1. Expenditures in period $t$ including transaction costs minus labor income:

$$p_t (c_t - n_t + T(c_t, m_t)).$$

Redefine gross consumption as: $x_t = c_t + T(c_t, m_t)$,

$=>$ Expenditures minus labor income in period $t$: $p_t (x_t(c_t, m_t) - n_t)$.
\[ x_t(c_t, m_t) = c_t + T(c_t, m_t) \] is increasing in \( c_t \). Invert function to \( c_t = f(x_t, m_t) \).

Replace \( c_t \) by \( f(x_t, m_t) \) to get indirect utility function:

\[
\max_{x(t), m(t)} V(0) = \int_0^\infty u[f(x(t), m(t)); 1 - n(t)] e^{-\rho t} dt \quad \text{s.t.} \quad \int_0^\infty x(t) e^{-\bar{\tau}(t) t} dt + \int_0^\infty [\pi(t) + r(t)] m(t) e^{-\bar{\tau}(t) t} dt = \frac{B_0}{p_0} + \frac{M_0}{p_0} + \int_0^\infty [w(t) - \tau(t)] e^{-\bar{\tau}(t) t} dt
\]

Formally equivalent to MIU approach with \( v(x(t), m(t), n(t)) \equiv u[f(x(t), m(t)); 1 - n(t)] \).

For exogenous labor supply \( n_t = y_t \):

\[ V(x_t, m_t) = U(c_t) = U(f(x_t, m_t)) \] , with \( x_t \) specified by: \( x_t = c_t + T(c_t, m_t) \).

Note that this is inconsistent with \( v \) being additive separable in \( x \) and \( m \).
3.4 Determination of the Price Level

Analyse questions like:
  a) Determine initial price level: money supply?
  b) Introduce government budget constraint
  c) Analyse conditions for superneutrality of money (growth model)
  d) Conditions for inflationary bubbles (Cagan model)
  e) Stochastic economy: Analyse dynamic reaction to shocks
  f) Introduce price rigidities (requires price setting)

Real effects of money are limited as long as prices are flexible. Here: look at determination of initial price level and bubbles

  → Bubbles and self fulfilling inflationary expectations
How to derive an explicit money demand function?

Consider CES utility function:

\[ U(C, \frac{M}{P}) = \left[ \alpha \left( \frac{C}{P} \right)^{\frac{1}{b}} + (1 - \alpha) \left( \frac{M}{P} \right)^{\frac{1}{b}} \right]^{b \cdot \frac{1}{b} - 1} \]

FOC => (1)

\[ \frac{U_{M/P}}{U_C} = \left( \frac{P_t C_t}{M_t} \cdot \frac{1 - \alpha}{\alpha} \right)^{1/b} = \frac{i_t - i_m}{1 + i_t} \]

Solving FOC gives money demand:

\[ \frac{M_t}{P_t} = \frac{1 - \alpha}{\alpha} \left( \frac{1 + i_t}{i_t - i_m} \right)^b C_t \]

Actually, this is not quite the LM-curve, because consumption in period t depends on the path of real balances.

Often, it is sufficient to have a relation between real balances and consumption (Example 1). Otherwise solve both FOCs simultaneously.

or: use additive separable utility function.
Example 1:
Closed economy: \( C_t = Y_t = F((K_t, N_t) \]

capital K and labor supply N given exogenously

\[ \Rightarrow \frac{M_t}{P_t} = \frac{1 - \alpha}{\alpha} \left( \frac{1 + i_t}{i_t - i_m} \right)^b Y_t = L(i_t)Y_t \]

\( \Leftrightarrow \) traditional LM-curve
Determinacy of the price level

Determine the price level for a given money supply process. Dynamic relation, because real money balances depend on the expected price path:

a) expectation about future monetary policy has impact on current money holding
b) possible: self fulfilling bubbles for price path

How to provide a nominal anchor?

Note: Fixing nominal rate of interest leaves price level indeterminate.

Take for example

\[ \frac{M_t}{P_t} = \frac{1 - \alpha}{\alpha} \left( \frac{1 + i_t}{i_t - i_m} \right)^b C_t \]

and consider a steady state for real economy: \( C_{t+1} = C_t = Y \) (with \( r = \rho \)).

Fixing \( i_t = i \) determines real money demand, but not the price level.
Fixing interest rate and money supply \( M_t \) determines \( P_t \).
3.4.1 Policy of constant money growth

FOC for real money balances can be reformulated as:

\[
\frac{U_m}{U_c} = \frac{i_t - i_m}{1 + i_t} \quad \Leftrightarrow \quad U_m = U_c \frac{i_t - i_m}{1 + i_t} \quad \Leftrightarrow
\]

\[
\left( u_c(C_t, m_t) - u_m(C_t, m_t) \right) = u_c(C_t, m_t) \left( 1 - \frac{i_t - i_m}{1 + i_t} \right) = u_c(C_t, m_t) \frac{1 + i_m}{1 + i_t}
\]

\[
= u_c(C_{t+1}, m_{t+1}) \frac{1 + r_t \cdot 1 + i_m}{1 + \rho \cdot 1 + i_t}
\]

\[
= u_c(C_{t+1}, m_{t+1}) \frac{1 + i_m}{1 + \rho} \cdot \frac{P_t}{P_{t+1}},
\]

where we exploit the Fisher equation. Setting \( i_m = 0 \), we get

\[
\left( u_c(C_t, m_t) - u_m(C_t, m_t) \right) \frac{1}{P_t} = \beta u_c(C_{t+1}, m_{t+1}) \frac{1}{P_{t+1}}
\]

\[
\Leftrightarrow \quad \left( u_c(C_t, m_t) - u_m(C_t, m_t) \right) \frac{M_t}{P_t} = \beta u_c(C_{t+1}, m_{t+1}) \frac{M_t}{P_{t+1}}
\]

where we exploit the Fisher equation. Setting \( i_m = 0 \), we get
Consider $M_{t+1} = (1 + \mu) M_t$. Rewrite FOC as:

$$ (u_c(C_t, m_t) - u_m(C_t, m_t)) m_t = \frac{\beta}{1 + \mu} u_c(C_{t+1}, m_{t+1}) m_{t+1} $$

(2)

Consider a steady state for real economy: $C_{t+1} = C_t = Y$ (with $r = \rho$).

Then the equilibrium is characterised by the difference equation:

(A) \[ F(m_t) = \frac{\beta}{1 + \mu} G(m_{t+1}) \]

with $F(m) = (u_c(Y, m) - u_m(Y, m)) m$ and $G(m) = u_c(Y, m) m$

and transversality condition $\lim_{T \to \infty} \frac{1}{(1 + r)^T} \frac{W_{T+1}^T}{P_T} = 0$. 

(B) TVC \( \lim_{T \to \infty} E_t(Q_{0,T+1}W_{T+1}) = 0 \) implies

\[
\lim_{T \to \infty} \beta^T G(m_T) = 0
\]

To see this, use Euler equation and recall that in the steady state \( r_i = r = \rho \) \( \forall t \). Hence:

\[
\beta^t G(m_t) = \beta^t U_c(c_t, m_t) \frac{M_t}{P_t} = \frac{1}{(1+r)^t} U_c(c_0, m_0) \frac{M_t}{P_t}.
\]

Since \( U_c(c_0, m_0) \) is a constant, we may normalize \( U_c(c_0, m_0) = 1 \), so that

\[
\beta^t G(m_t) = \frac{1}{(1+r)^t} \cdot \frac{M_t}{P_t} = \frac{(1+\mu)^t}{(1+r)^t(1+\pi)^t} \cdot \frac{M_0}{P_0} = \left(\frac{1+\mu}{1+i}\right)^t \cdot \frac{M_0}{P_0}.
\]

Which converges for \( \mu < i \).

Thus, TVC implies \( \mu < i \).
For additive separable utility $u(C, m) = u(C) + V(m)$, we have:

$$F(m) = (u_c(Y) - V_m(m))m \quad \text{and} \quad G(m) = u_c(Y)m.$$ 

In a steady state equilibrium: $F(m) = \frac{\beta}{1 + \mu} G(m)$.

$m > 0$ implies $u_c > V_m$. Furthermore, $V'' < 0$. Thus,

$$F'(m) = [u_c - V_m(m)] - mV''(m) > 0.$$ 

Outside the steady state, $F'$ may be negative. $F'' = -m V''' - 2V''$. Sign depends on $V'''$.

**A steady state** $m^*$ solves

$$\left( u_c(Y) - V_m(m) \right) m = \frac{\beta}{1 + \mu} u_c(Y)m. \quad (3)$$

Suppose, $\lim_{m \to 0} m \cdot V_m(m) = 0$, or more restrictive $V_m(0) < \infty$.

Then, there exists a solution $m^* = 0$, => non-monetary steady state.
Is there a monetary steady state with $m^* > 0$?

$$
(u_c(Y) - V_m(m)) = \frac{\beta}{1 + \mu} u_c(Y) \iff 1 - \frac{V_m(m^*)}{u_c(Y)} = \frac{\beta}{1 + \mu} \iff \frac{V_m(m^*)}{u_c(Y)} = 1 - \frac{\beta}{1 + \mu}.
$$

$LHS > 0$, $LHS$ decreasing in $m$.

Provided that $\lim_{m \to 0} V_m(m)$ is sufficiently large and $\lim_{m \to \infty} V_m(m) = 0$, a positive solution $m^*$ exists if

$$
RHS > 0 \iff \beta < 1 + \mu \iff \mu \geq \beta - 1 = -\frac{\rho}{1 + \rho}.
$$

(note: $\beta = \frac{1}{1 + \rho}$)

Necessary condition for existence of mon. steady state.

E.g., for $\mu = -\frac{\rho}{1 + \rho}$, $m^* = \bar{m}$ with $V_m(\bar{m}) = 0$ (satiation level)

negative growth rate of money supply $\to$ deflation [zero nominal interest rate, Friedman rule]

**Note**: monetary steady state $m^*$ is unstable, because $V''(m(t)) < 0$, so $\Delta(m_t) > 0$ for $m > m^*$
With forward looking behaviour (rational expectations), price level $p_0$ and price path $p(t) = p_0 e^{(\mu - \nu) t}$ are pinned down uniquely if the unstable paths violate some constraints!

Key: What happens to $\lim_{m(t) \to 0} m(t) V_m(m(t))$?

Bubbles are ruled out if $\lim_{m(t) \to 0} m(t) V'(m(t)) > 0 \Rightarrow V'(m) \to \infty$

Money must be essential: Marginal utility of money increases faster than the rate at which money goes to zero.
Specific payoff functions:

A) CES payoff function \[ V(m) = \frac{1}{1-1/\sigma} m^{1-1/\sigma} \Rightarrow V'(m) = m^{-1/\sigma} \]

\[ \lim_{m \to 0} mV'(m) = \lim_{m \to 0} m^{1-1/\sigma} > 0 \text{ for } \sigma < 1 \]

law of motion from difference equation (A) and normalization \[ u'(C_0) = 1 \text{ yield} \]

\[
m_t u'(C_t) - m_t V'(m_t) = \frac{\beta}{1 + \mu} u'(C_{t+1}) m_{t+1}.
\]

(A), (3)

\[
m_t u'(C_t) - m_t^{-1/\sigma} = \frac{\beta}{1 + \mu} u'(C_{t+1}) m_{t+1}
\]

For constant \( C \),

\[ \Rightarrow \frac{\dot{m}}{m} = (\mu + \rho) \left(1 - m^{-1/\sigma}\right) \]
B) Indirect utility for the Cagan demand function:

\[ \frac{M_t^d}{P_t} = C (1 + i_t)^{-b} \]

Cagan money demand:

\[ \Rightarrow \ln m = \ln M - \ln P = c - b \ln(1 + i) \approx c - b i \quad \Leftrightarrow \quad i = \frac{c - \ln m}{b} \]

Normalize \( u'(C_t) = 1 \) , Sidrauski FOC \( \Rightarrow \) \( V'(m) = \frac{i}{1 + i} \approx i = \frac{c - \ln m}{b} > 0 \) for \( c > \ln m \)

required payoff function: \( V(m) = m \frac{c - \ln m + 1}{b} \).

Note that \( V''(m) = -\frac{1}{b m} < 0 \).

\[ \lim_{m \to 0} m V'(m) = -\lim_{m \to 0} m \ln m / b = 0. \quad \text{Money is not essential} \]

Cagan (discrete time): \( \ln M_t - p_t = c - b \left( E p_{t+1} - p_t \right) \).

Continuous time: \( \ln M(t) - p(t) = c - b \dot{p}(t) \).
Continuous time: \[ \ln M(t) - p(t) = c - b \dot{p}(t) . \]

For constant rate of money growth rate \( \mu \):
\[ \dot{p}(t) = \frac{1}{b} [\ln P(t) + c - \ln M_0 \mu t] , \]
thus for steady state: \( P(t) = C M_0 e^{\mu t} = C M(t) \)

Saddle-point stability.
Since Money is not essential, self fulfilling explosive price paths are not ruled out here.
3.4.2 Policy of a pure interest rate peg (Sargent/Wallace)

Fix nominal interest rate: \( \{i_t = i + u_t\} \forall t \) with \( E(u_t) = 0 \)

Money supply is endogenous

a) All real variables are well specified: Real balances are uniquely determined by FOC:
\[
V_m(m) = \frac{i}{1+i}
\]

b) Rate of inflation is uniquely determined by Fisher equation:
\[
E(p_{t+1}|t) - p_t = E(\pi) = E(i) - r
\]

   No speculative bubble for the price path.

But: Initial price level \( P_0 \) is not specified - there is nominal indeterminacy:

If \( M_0 \) (with \( P_0 \)) is a solution to \( V_m(m_0) = \frac{i}{1+i} \), then also any \( \lambda M_0 \) (with \( \lambda P_0 \)).

Difference equation: \( [E(p_{t+1}|t) - p_t] = i - r \) = constant

not sufficient to pin down the initial price level:

If \( \ln P_t \) is a solution (with \( \ln M_t \)), then also \( \ln P_t + \alpha \) with \( \ln M_t + \alpha \)

**Key Lesson**: Pure interest rate peg leaves price level indeterminate.
In contrast, feedback interest rate rules are able to control price level:  

3.4.3 Interest Rate Feedback Rules

Example: Inflation Targeting/ or Price Level Targeting

Announced (Credible) Central Bank Rule:

\[ i_t = \bar{i} + f \cdot (\pi_t - \pi^*) + u_t \]  

(cf. Taylor rule)

Under what conditions do we get a determinate price level?

Expected interest rate:

\[ E_{t-1}i_t = \bar{i} + f (E_{t-1}\pi_t - \pi^*) + E_{t-1}u_t \]

Implied dynamics of inflation, using Fisher equation \( E_{t-1}(i_t) = \bar{\rho} + E_{t-1}(\pi_{t+1}) \):

\[ E_{t-1}\pi_{t+1} = \bar{i} - \bar{\rho} + f (E_{t-1}\pi_t - \pi^*) + E_{t-1}(u_t) \]  

(1)

First order difference equation for inflation.
\[ E_{t-1}\pi_{t+1} = \bar{\pi} - \bar{\pi} + f(E_{t-1}\pi_t - \pi^*) + E_{t-1}(u_t) \]  

Forward Looking Solution:

\[ \pi_t = \pi^* + \frac{1}{f}(E\pi_{t+1} - (\bar{\pi} - \bar{\pi})) \]

Analyse perfect foresight solution, using

\[ \bar{\pi} - \bar{\pi} = \pi^*; E_{t-1}(u_t) = 0: \quad (1) \Rightarrow \; \pi_{t+1} - \pi^* = f(\pi_t - \pi^*) \]

\[ \pi_t = \pi^* + \frac{1}{f^T}(\pi_{t+T} - \pi^*) \iff \pi_{t+T} - \pi^* = f^T(\pi_t - \pi^*) \]

\( f > 1 \), uniquely determined dynamics of inflation provided that we rule out bubbles!
In a stochastic system, always reversal to \( \pi^* \) after deviations from the inflation target: \( \pi_t \rightarrow \pi^* \) !

For \( f < 1 \), infinite number of solutions depending on expected inflation \( E(\pi_{t+T}) \)
Interest rate rules

\[ \pi_{t+1} - \pi^* = f(\pi_t - \pi^*) \]

f<1: solution indeterminate
For $f > 1$, the price level is also uniquely determined. Using $\pi_t = P_t - P_{t-1}$, we get the second order difference equation $p_{t+1} = p_t + \pi^* + f(p_t - p_{t-1} - \pi^*)$.

In period $t$, $p_{t-1}$ is given. $\Rightarrow p_t$ is determinate if $f > 1$. 