Advanced Macroeconomic Analysis 2
Part I: Monetary Macroeconomics

Overview
1. Money and Inflation - The Cagan Model of Money and Prices
2. Overlapping-Generations Model
3. Money in the Utility Function (Brock/Sidrauski Model, Cash-in-advance model, shopping-time model)
4. Monopolistic Competition and Price Rigidities
5. The New Neoclassical Synthesis and Monetary Policy
6. Commitment and Discretion in Monetary Policy
7. Stabilizing Demand and Supply Shocks
8. Current Developments and Open Issues
1. Money and Inflation

**Basic Questions**

**Foundation of Money:**
Why do we use money?
What is the origin of welfare gains in a monetary economy?

**Positive questions:**
How is inflation related to money supply, government debt and information about the state of economy?
What are the transmission mechanisms of monetary policy?
How do prices adjust to shocks (shocks of real economy, asset markets, monetary policy)?

**Normative questions:**
What is the adequate design of monetary policy?
What is the appropriate task of central banks?
How do we define price stability?
Is there a role for active stabilization policy?
Classical view:
Neutrality of monetary policy, irrelevance due to rational expectations
There is no role for money if private agents are rational and forward looking.
Real Business Cycle Approach neglects the role of money.

Modern New Neoclassical Synthesis:
Sticky prices as key feature motivate a positively sloped AS curve.
Intertemporal optimisation motivates an IS / AD curve: forward looking agents.
LM curve motivated by central bank behaviour.
Frictionless economy serves merely as reference point: efficient markets.
1.1. **Money and Inflation – The Cagan Model of Money and Prices** (Cagan, 1956)

Slides partially adopted from G. Illing, University of Munich.

First, we start with an old fashioned approach – just assuming there is a demand for money! We ask: **how is the price path determined by the expected money supply process?** The answer follows from equilibrium with forward looking agents.

**Basics – Fisher equation under uncertainty:**

\[
(1 + i_t) = (1 + r_t) \frac{E P_{t+1}}{P_t} = (1 + r_t) (1 + E(\pi_t)) \approx 1 + r_t + E(\pi_t)
\]

Log-approximation: \( \ln(1+x) \sim x \)

\[
1 + \pi_t = \frac{P_{t+1}}{P_t} \quad \Rightarrow \quad \pi_t \approx \ln(P_{t+1}) - \ln(P_t)
\]

\[
=> \quad i_t \approx r_t + E(\pi_t) \approx r_t + E \left( \ln(P_{t+1}) \right) - \ln(P_t)
\]
Cagan’s isoelastic money demand function (allows linear analysis)

Discrete version \( \frac{M_t^d}{P_t} = C (1+i_t)^{-b} \).

Compare with traditional LM curve: \( \frac{M_t^d}{P_t} = L(Y_t,i_t) \)

Normalizing \( C=\left(\frac{1}{1+r}\right)^{-b} \) gives \( \frac{M_t^d}{P_t} = \left(\frac{1+i_t}{1+r}\right)^{-b} = \left(\frac{E P_{t+1}}{P_t}\right)^{-b} \) or in logarithmic terms:

\[
m_t^d - p_t = -b \left[ \ln(E(P_{t+1})) - \ln(P_t) \right] \approx -b \left( E \ p_{t+1} - p_t \right)
\]

Determination of the Price Level for a given stochastic money supply path \( \{M_{t+s}\} s=0,…,T \):  
**Equilibrium** on the money market: \( m_t^d = m_t \); gives linear difference equation:

\[
m_t - p_t = -b \left( E \ p_{t+1} - p_t \right) \Leftrightarrow p_t = \frac{1}{1+b} m_t + \frac{b}{1+b} E p_{t+1} \quad \text{forward looking solution}
\]
\[ p_t = \frac{1}{1+b} m_t + \frac{b}{1+b} E p_{t+1} \]

Price level as forward looking variable (non predetermined)

Substitute \( E_t p_{t+1} = \frac{1}{1+b} E_t m_{t+1} + \frac{b}{1+b} E_t E_{t+1} p_{t+2} \)

Use law of iterated expectations \( E_t E_{t+1} p_{t+2} = E_t p_{t+2} \) to get:

\[ p_t = \frac{1}{1+b} (m_t + \frac{b}{1+b} E_t (m_{t+1})) + \left( \frac{b}{1+b} \right)^2 E_t p_{t+2} \]

Iterated substitution gives:

\[ p_t = \frac{1}{1+b} \sum_{s=0}^{T-1} \left( \frac{b}{1+b} \right)^s E(m_{t+s}) + \left( \frac{b}{1+b} \right)^T E p_{t+T} \]

Assume that the sum converges \( \lim_{T \to \infty} \sum_{s=0}^{T-1} \left( \frac{b}{1+b} \right)^s E(m_{t+s}) < \infty \)
\[ p_t = \frac{1}{1+b} \sum_{s=0}^{\infty} \left( \frac{b}{1+b} \right)^s E(m_{t+s}) + \lim_{s \to \infty} \left( \frac{b}{1+b} \right)^s E p_{t+s} \]

Rule out hyperinflationary bubbles:

If we impose restriction the \( \lim_{s \to \infty} \left( \frac{b}{1+b} \right)^s E p_{t+s} = 0 \), then

\[ \Rightarrow \text{ solution: } p_t^F = \frac{1}{1+b} \sum_{s=0}^{\infty} \left( \frac{b}{1+b} \right)^s E(m_{t+s}) \]

So, the price level is determined by expectations about “future fundamentals”
Specify expected future money supply process!

Note: \[ \sum_{s=0}^{\infty} \left( \frac{b}{1+b} \right)^s = 1 + b \; ; \quad \sum_{s=0}^{\infty} s \left( \frac{b}{1+b} \right)^s = b(1+b) \]

Examples: see also Homework!

A) Constant money supply: \( m_{t+s} = m_t = m \)

\[ p_t = \frac{1}{1+b} \sum_{s=0}^{\infty} \left( \frac{b}{1+b} \right)^s m_t = m_t \]
B) Constant growth rate of money:

\[ M_{t+s} = M_t \cdot (1 + \mu)^s \quad \Leftrightarrow \quad m_{t+s} = m_t + \mu s \]

\[ p_t = \frac{1}{1+b} \sum_{s=0}^{\infty} \left( \frac{b}{1+b} \right)^s m_{t+s} = \frac{1}{1+b} \sum_{s=0}^{\infty} \left( \frac{b}{1+b} \right)^s (m_t + s\mu) \]

\[ = \frac{1}{1+b} \sum_{s=0}^{\infty} \left( \frac{b}{1+b} \right)^s m_t + \frac{1}{1+b} \sum_{s=0}^{\infty} s \left( \frac{b}{1+b} \right)^s \mu = m_t + b\mu \]

\[ \Rightarrow p_{t+1} - p_t = m_{t+1} - m_t = \mu \quad \Rightarrow \quad \pi = \mu \]

C) Anticipated increase in money supply at some future date \( t+T \) (\( T \) periods from now):

\[ m_{t+s} = \begin{cases} m_1 & s < T \\ m_2 > m_1 & s \geq T \end{cases} \]

\[ p_t = \frac{1}{1+b} \sum_{s=0}^{T-1} \left( \frac{b}{1+b} \right)^s m_1 + \frac{1}{1+b} \sum_{s=T}^{\infty} \left( \frac{b}{1+b} \right)^s m_2 \]

\[ p_t = \frac{1}{1+b} \sum_{s=0}^{\infty} \left( \frac{b}{1+b} \right)^s m_1 + \frac{1}{1+b} \sum_{s=T}^{\infty} \left( \frac{b}{1+b} \right)^s (m_2 - m_1) \]
\[ p_t = \frac{1}{1+b} \sum_{s=0}^{\infty} \left( \frac{b}{1+b} \right)^s m_1 + \frac{1}{1+b} \sum_{s=T}^{\infty} \left( \frac{b}{1+b} \right)^s (m_2 - m_1) \]

\[ p_t = m_1 + \frac{1}{1+b} \left( \frac{b}{1+b} \right)^T \sum_{s=0}^{\infty} \left( \frac{b}{1+b} \right)^s (m_2 - m_1) = m_1 + \left( \frac{b}{1+b} \right)^T (m_2 - m_1) \]

\[ p_{t+1} = m_1 + \left( \frac{b}{1+b} \right)^{T-1} (m_2 - m_1) \quad \text{provided} \ T \geq 1 \]

\[ \Rightarrow \pi_t = p_{t+1} - p_t = \left[ \left( \frac{b}{1+b} \right)^{T-1} - 1 \right] \left( \frac{b}{1+b} \right)^T (m_2 - m_1) = \frac{1}{b} \left( \frac{b}{1+b} \right)^T (m_2 - m_1) \]

Anticipated temporary increase in money supply at some future date \( t+T \) leads to an increase in the price level today.

The further in the future money supply rises, the smaller its effect on today’s price level.

General Intuition: Assume there is an expected increase in the money stock at \( t+T \). Then \( P_{t+T} \) must rise to clear the market. Consequently, expected inflation in the period before, that is \( t+T-1 \), is higher. This reduces the demand for money in period \( t+T-1 \), so \( P_{t+T-1} \) must rise to clear the market.

\( \Rightarrow \) Expected inflation in \( t+T-2 \) rises, dampening the money demand, and so on:

Consequently: direct impact today (discounted with the rate \( b/1+b \))
D) **Stochastic money supply with AR(1) process:**

\[ m_{t+1} = \rho m_t + \varepsilon_{t+1}; \quad 0 \leq \rho \leq 1; \quad E_t \varepsilon_{t+1} = 0; \quad E_t (\varepsilon_{t+s} \varepsilon_{t+s'}) = 0 \quad \text{for} \ s \neq s'. \]

So \( E_t m_{t+s} = \rho^s m_t \)

So \( p_t = \frac{1}{1+b} \sum_{s=0}^{\infty} \left( \frac{b \rho}{1+b} \right)^s m_t = \frac{1}{1+b} \sum_{s=0}^{\infty} \left( \frac{b \rho}{1+b} \right)^s m_t = \frac{1}{1+b - \rho b} m_t \)

\( p_{t+1} = \frac{1}{1+b - \rho b} m_{t+1} = \frac{1}{1+b - \rho b} (\rho m_t + \varepsilon_{t+1}) = \rho \frac{1}{1+b - \rho b} m_t + \frac{\varepsilon_{t+1}}{1+b - \rho b} \)

\( \Rightarrow \) Prices are an AR(1) process.

(i) \( m_t \) follows random walk (\( \rho = 1 \)) \( \Rightarrow \) \( p_t = m_t \)

(ii) \( m_t \) serially uncorrelated: (\( \rho = 0 \)) \( \Rightarrow \) \( p_t = \frac{1}{1+b} m_t = \frac{1}{1+b} \varepsilon_t \)
Open issues:

a) Rationale of money supply process
b) Rationale of ruling out hyperinflationary bubbles
c) Rationale of money demand (microfoundation for LM curve and Cagan demand function)

A) Rationale of Monetary Policy: Create Seigniorage
(hyperinflation as consequence of the need to finance government spending)

Seigniorage from money creation: \( S_t = \frac{M_t - M_{t-1}}{P_t} = \frac{M_t - M_{t-1}}{M_t} \cdot \frac{M_t}{P_t} \)

Steady state government revenue from money creation?

With constant money growth rate, \( M_t = (1 + \mu) M_{t-1} \), seigniorage is: \( S_t = \frac{\mu}{1 + \mu} \cdot \frac{M_t}{P_t} \)

**Cagan**: Isoelastic Money demand gives \( \frac{M_t}{P_t} = \left( \frac{P_{t+1}}{P_t} \right)^{-b} \)
Constant process of money growth: $M_{t+1} = (1 + \mu) M_t$ generates: $P_{t+1} = (1 + \mu) P_t$ so:

$$S = \frac{\mu}{1 + \mu} (1 + \mu)^{-b} = \mu (1 + \mu)^{-(1+b)}.$$ 

Maximum seigniorage at: $\frac{\partial}{\partial \mu} S = 0$:

$$\frac{\partial S}{\partial \mu} = (1 + \mu)^{-(1+b)} - (1 + b) \mu (1 + \mu)^{-(2+b)} = 0$$

\[ \Leftrightarrow (1 + \mu) = (1 + b) \mu \Leftrightarrow \mu = \frac{1}{b} \]

Equivalently (continuous time analysis):

$$\frac{M_t}{P_t} = e^{-b \pi_r}, S_t = \pi_t e^{-b \pi_t}$$

Optimal supply for monopolist without cost $\mu b = 1$

Puzzle:

Empirical estimates of elasticity: $1/b \sim$ at most around 10-30% per month. Why do we observe hyperinflations with higher growth rates of money supply?

a) Adaptive expectations (Cagan): lag in reaction of money demand **Homework!**

b) Incentives for surprise inflation (debt relief) – problem of dynamic consistency
B) Bubbles

Forward looking solution gives first order difference equation:

\[ p_t = \frac{1}{1+b} m_t + \frac{b}{1+b} E \, p_{t+1} \]

One solution of this difference equation is:

\[ p_t^F = \frac{1}{1+b} \sum_{s=0}^{\infty} \left( \frac{b}{1+b} \right)^s E(m_{t+s}) \]

Without additional anchor, that difference equation has many other solutions (not specified without some endpoint constraint):

\[ p_t^y = p_t^F + y_t = \frac{1}{1+b} m_t + \frac{b}{1+b} E \left( p_{t+1}^F + y_{t+1} \right) \]

\[ \Rightarrow \quad y_t = \frac{b}{1+b} E \left( y_{t+1} \right) \]
Assume a stochastic process \( \{y_t\} \):
\[
y_{t+1} = \frac{b+1}{b} y_t + v_{t+1} \quad \text{with } E_t v_{t+1} = 0, \text{ then } y_t = \frac{b}{1+b} E_t y_{t+1},
\]
and \( y^F_t = p_t^F + y_t \) is also a solution.

\( y_t \) is a bubble, with \( y_{t+s} \) exploding at the rate \( 1+1/b \).

\( p_t = p^F_t \) is unique solution only if the no-bubble condition holds:
\[
\lim_{s \to x} \left( \frac{b}{1+b} \right)^s E p_{t+s} = 0.
\]

Economic Motivation?
Self fulfilling inflationary expectations (erosion of trust, “speculative hyperinflation”) may make money useless as a medium of exchange without violating aggregate resource constraints!

\[\Rightarrow \] Money has to be essential to rule out a hyperinflationary price path!

\[\Rightarrow \textbf{need of a microeconomic foundations} \] for the money demand function!
Appendix: Some further approximation rules

1. Consider some stochastic variable $P$ with small variance.

Then $E(\ln(P)) \approx \ln(E(P))$

Proof: Suppose $P$ has a log-normal distribution, i.e.

$$p = \ln(P) \sim N(\mu, \sigma^2).$$

Then, $E(P) = E(e^p) = e^{\mu+\sigma^2/2}$.

Hence, $\ln(E(P)) = \mu + \sigma^2 / 2 \approx \mu$ for small Variance $\sigma^2$

For diminishing variance all distributions converge to a one-point distribution. Hence, the approximation also holds for any other distribution. QED
2. $n^{th}$ order Taylor approximation:

$$f(a + x) \approx f(a) + f'(a) \frac{x}{1!} + f''(a) \frac{x}{2!} + \cdots + f^{(n)}(a) \frac{x^n}{n!}$$

Let $f : D \subseteq \mathbb{R}^p \to \mathbb{R}$ be a continuous function, and suppose that $f$ has continuous partial derivatives of order $n$ in a neighbourhood of every point on a line segment $[a, b] \subseteq D$. Then there exists a point $\tilde{a} \in [a, b]$ such that for $\tilde{a} = a + x$ the above approximation holds with equality.